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Buckling and vibration of a rotating spoke

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SUMMARY
The buckling and vibration of the spoke of a rotating wheel is examined.

Since the axial load is a function of position closed form solutions for the eigenmodes are proscribed and recourse is made to regular and singular perturbation expansions in terms of several dimensionless parameters appearing in the governing equations. Some numerical results are also included in the interest of completeness.

1. Introduction

The increased importance attached to energy storage has given an impetus to the study and design of such devices as flywheels (see Figure 1).

Operating at high rotation speeds, as they normally do, flywheels exert potentially destablizing stresses on their structural components. In particular, the spokes of such a flywheel experience a compressive axial stress near the rim which is proportional to the

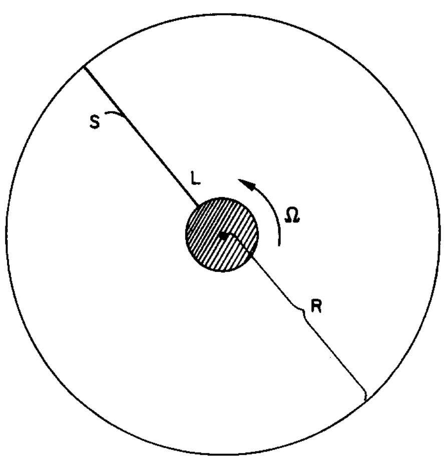

Figure 1.

square of the angular velocity and hence buckled states could be maintained at sufficiently high operating speeds. In addition, the investigation of the free vibration of these spokes is clearly of importance in conjunction with avoiding resonance. It is our intention to examine both of these topics in this paper.

As will be seen in the next sections the governing static and dynamic equations preclude closed form solutions. However, the dimensionless versions of these equations contain small parameters so that asymptotic expansions of the eigenmodes and eigenvalues in terms of the relevant parameter may be contemplated.

The regular perturbation problem associated with the static buckling, Section 2, is reminiscent of an earlier investigation by Nachman [1] for rods clamped at the rim only. The complications attendant with the axial stress in the present problem, however, (compressive near the rim and tensile near the hub as opposed to purely compressive in the rim-clamped case) prevent us from achieving even the modest degree of compactness of presentation to be found in the earlier work.

In the vibration analysis, Section 3, attention is restricted to high rotation velocities (in consonance with present technological interests) and the result is a singular perturbation problem with the added complication of a turning point located within the interval of interest. A uniformly valid solution is constructed and approximations to the eigenfrequencies deduced. Earlier work of a related nature was done by Lakin ([2], [3]), Laking and Ng [4] and Lakin and Nachman [5].

A modest numerical section is also included to augment the analytical work.

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2. Static buckling

An elastic rod of length L is clamped to the hub and rim of a steadily rotating wheel of radius R, as in Figure 1. When the angular velocity of the wheel is Ω the axial stress in the unbuckled rod is given by

$$T = \rho A\Omega^2\{\tfrac{1}{2}L(R - \tfrac{2}{3}L) + (L - s)[\tfrac{1}{2}(L + s) - R]\}, \tag{1.1}$$

where s is arclength along the rod measured from the rim, ρ is the density of the rod and A is its cross-sectional area.

Since $L < R$ ($R - L =$ hub radius) we have that

$$T(0) = \tfrac{1}{6}\rho A\Omega^2 L(L - 3R) < 0: \quad \text{compression,}$$

$$T(L) = \tfrac{1}{2}\rho A\Omega^2 L(R - \tfrac{2}{3}L) > 0: \quad \text{tension.}$$

The existence of compressive loads in any part of the rod implies the possible corresponding existence of static buckled states while the simultaneous existence of tensile loads in the remainder of the rod is the novel facet of this problem and is the source of the mathematical difficulties.

If we let EI represent the flexural rigidity of the rod, a balance of forces and moments results in the equation

$$EI\frac{d^4y}{ds^4} - \frac{d}{ds}\left(T\frac{dy}{ds}\right) = \nu\rho A\Omega^2 y, \tag{1.2}$$

where $y(s)$ is the small deflection in (out of) the plane of rotation for $\nu = 1(0)$. Setting $x = s/L$, $\lambda = \rho AL^4\Omega^2/EI$, $\alpha = R/L$ and substituting expression (1.1) for T we obtain

$$y^{(4)} = \lambda\{[f(x, \alpha)y']' + \nu y\} \tag{1.3}$$

where the prime is understood to be d/dx and

$$f(x, \alpha) = \tfrac{1}{2}(\alpha - \tfrac{2}{3}) + (1 - x)[\tfrac{1}{2}(1 + x) - \alpha], \qquad \alpha \geq 1 \tag{1.4}$$

is a dimensionless axial stress. The boundary conditions corresponding to clamped ends at $s = 0, L$ are

$$y(0) = y'(0) = y(1) = y'(1) = 0. \tag{1.5a–d}$$

Case I. $\nu = 0$

Integrating (3) from 0 to x, setting $k = y'''(0)$ and $v(x) = y'(x)$ we arrive at

$$v'' - \lambda f(x, \alpha)v = k, \tag{1.6}$$

$$v(0) = v(1) = \int_0^1 v(x)dx = 0. \tag{1.7a–c}$$

Equations (1.6) and (1.7) represent a rather unusual eigenvalue problem. Nonhomogeneous Sturm–Liouville problems subject to three homogeneous constraints have turned up in [6] and doubtless elsewhere. The more serious question resides in the fact that the density, or weight function, $f(x, \alpha)$ is not of one sign on the interval of interest. We will have more to say about this shortly.

It proves convenient to make the identification

$$w(x) = v(x) - x(x - 1)k/2. \tag{1.8}$$

Then

$$w'' - \lambda f(x, \alpha)w = f(x, \alpha)x(x - 1)\lambda k/2, \tag{1.9}$$

$$w(0) = w(1) = 0, \quad \int_0^1 w(x)dx = k/12. \tag{1.10a–c}$$

Consider next the eigenvalue problem

$$\phi''(x) - \mu f(x, \alpha)\phi(x) = 0, \quad \phi(0) = \phi(1) = 0. \tag{*}$$

There is no formal difficulty in producing explicit solutions of (*). In fact, if we set

$$z = (2\mu)^{\frac{1}{3}}(x - \alpha)$$

then (*) is transformed into

$$\phi_{zz} + [\tfrac{1}{4}z^2 - a]\phi = 0, \quad a = \tfrac{1}{2}\sqrt{\mu/2}\,(\alpha^2 - \alpha + \tfrac{1}{3}), \tag{1.11a–b}$$

$$\phi(-(2\mu)^{\frac{1}{3}}\alpha) = \phi((2\mu)^{\frac{1}{3}}(1 - \alpha)) = 0, \tag{1.12}$$

so that $\phi(z)$ is a linear combination of the parabolic cylinder functions $W(a, z)$ and $W(a, -z)$, [7]. The eigenvalues μ_n follow from imposing the boundary conditions (1.12a–b); the negative ones are most easily obtained by replacing a with $-ia$ and z with $e^{i/4}z$.

In what follows it is necessary to know that the ϕ_n's are complete over some suitable space. The fact that $f(x, \alpha)$ changes sign on [0, 1] means that (*) is excluded from the theorems on completeness developed for standard Sturm–Liouville problems. The proof that $\{\phi_n(x)\}$ is a complete, orthonormal set is contained in [8]. It suffices here to say that the ϕ_n's are orthonormal with respect to the weight $-\mu_n f(x, \alpha)$ and any function which is twice differentiable and equals zero at $x = 0, 1$ is contained in the space spanned by $\{\phi_n(x)\}$.

Thus we may write

$$w(x) = \sum_{-\infty}^{\infty} w_n\phi_n(x), \quad kx(x - 1)/2 = \sum_{-\infty}^{\infty} k_n\phi_n(x),$$

where

$$w_n = -\mu_n \int_0^1 w(x)f(x, \alpha)\phi_n(x)dx,$$

$$\begin{aligned} k_n &= -\mu_n \int_0^1 \frac{k}{2}x(x - 1)f(x, \alpha)\phi_n(x)dx \\ &= -\mu_n \int_0^1 \phi_n(x)[w''/\lambda - f(x, \alpha)w]dx = \frac{\mu_n - \lambda}{\lambda} w_n. \end{aligned}$$

Thus

$$w(x) = k \sum_{-\infty}^{\infty} \frac{\lambda}{\lambda - \mu_n}\left(\int_0^1 \phi_n(x)dx\right)\phi_n(x).$$

Enforcing condition (1.10c) results in the eigenvalue equation

$$\tfrac{1}{12} = \sum_{-\infty}^{\infty} \frac{\lambda}{\lambda - \mu_n}\left(\int_0^1 \phi_n(x)dx\right)^2 = F(\lambda) \tag{E1}$$

which, by Figure 2, defines a countably infinite set of eigenvalues λ_n such that $\mu_n < \lambda_n < \mu_{n+1}$.

Once a suitable normalization has been adopted (e.g., $k = 1$ or $\int_0^1 v_n^2 dx = 1$) an attempt to graph the buckled modes $y_n(x) = \int_0^x v_n(\xi)d\xi$ might be initiated. We instead content ourselves with the observation that, with $\lambda_n > 0$, the solutions to (1.6)–(1.7) may be written

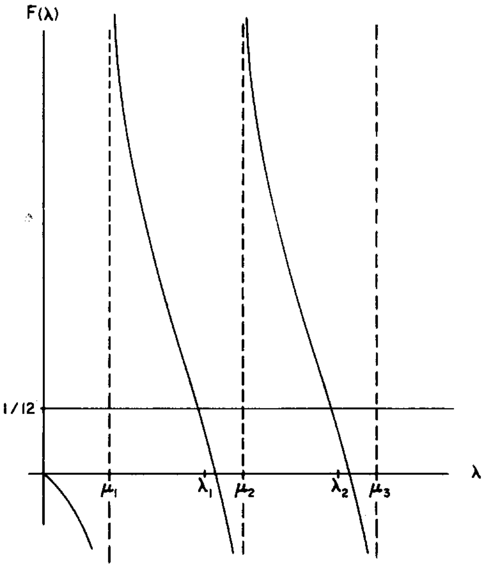

Figure 2.

$$V_n(x) = A_n W(\gamma_n, x) + B_n W(\gamma_n, -x) + k\left\{W(\gamma_n, -x)\int_0^x W(\gamma_n, t)dt - W(\gamma_n, x)\int_0^x W(\gamma_n - t)dt\right\},$$

where $\gamma_n = \frac{1}{2}\sqrt{\lambda_n/2}\,(\alpha^2 - \alpha + \frac{1}{3})$. While this expression is not well suited for discussing the existence and distribution of the λ_n, or we would have used it immediately after (1.7), it does allow us to see that $v_n(x)$ and hence $y_n(x)$ "wiggles" only for $0 < x < \alpha - \sqrt{\alpha^2 - \alpha + \frac{1}{3}}$, precisely where $f(x, \alpha) < 0$.

Some numerical work augmenting (E1) and Figure 2 is provided in Section 4.

Case II. $\nu = 1$

While Case I could be handled without much concern about α other than its being ≥ 1, the same will not be true here. Specifically, Equation (3) cannot be integrated and some approximation scheme must be used if any sort of analytical expressions are desired. We therefore examine the eigenvalue problem for $\alpha \gg 1$ which entails asymptotic expansions of λ and y in terms of powers of $\varepsilon = 1/\alpha$, and the subsequent solution of a regular perturbation problem.

If we set $1/\alpha = \varepsilon$, Equation (3) assumes the form

$$\varepsilon y^{(4)} = \lambda\{(\langle\tfrac{1}{2}(1 - \tfrac{2}{3}\varepsilon) + (1 - x)[\tfrac{1}{2}\varepsilon(1 + x) - 1]\rangle y')' + \varepsilon y\}. \tag{1.13}$$

While Equation (1.13) has all the earmarks of a singular perturbation problem it must be kept in mind that $\lambda = \lambda(\varepsilon)$ and hence the distinguished limit as $\varepsilon \to 0$ must be deduced. There are essentially three possibilities.

1. $\lim_{\varepsilon\to 0} \varepsilon/\lambda = \infty$: Clearly here we would have $y^{(4)} \sim 0$ which, together with the boundary conditions (5a–d), implies $y \sim 0$.

2. $\lim_{\varepsilon \to 0} \varepsilon/\lambda = 0$: This option takes a little untangling before it can be discarded. The interval $[0, 1]$ must be split into two main regions and three boundary layer regions. There are boundary layers at $x = 0, 1$ and around $x = \frac{1}{2}$, which is a turning point under the above limiting process. The two main regions occupy the spaces between the boundary layers. The boundary layer at $x = 0$ has $\xi_L = (\varepsilon/\lambda)^{-\frac{1}{4}} x$ as its independent variable and

$$\frac{d^4 y_L}{d\xi_L^4} + \frac{1}{2} \frac{d^2 y_L}{d\xi_L^2} = 0$$

as its governing differential equation. Similarly the boundary layer at $x = 1$ is described in terms of $\xi_R = (\varepsilon/\lambda)^{-\frac{1}{4}}(1 - x)$ and

$$\frac{d^4 y_R}{d\xi_R^4} - \frac{1}{2} \frac{d^2 y_R}{d\xi_R^2} = 0$$

and the boundary layer at $x = \frac{1}{2}$ is described in terms of $\xi_M = (\varepsilon/\lambda)^{-\frac{1}{3}}(x - \frac{1}{2})$ and

$$\frac{d^4 y_M}{d\xi_M^4} - \frac{d}{d\xi_M}\left(\xi_M \frac{dy_M}{d\xi_M}\right) = 0.$$

The two main regions are governed by $[(x - \frac{1}{2})y']' = 0$. Applying the boundary conditions and matching results in $y \sim 0$.

We may therefore conclude that $\lambda = \varepsilon \Lambda$ and adopt the expansions

$$\Lambda \sim \sum_{n=0}^{\infty} \Lambda_n \varepsilon^n, \quad y \sim \sum_{n=0}^{\infty} y_n(x) \varepsilon^n$$

which, when substituted into (1.13) and (1.5a–d), results in the following $O(1)$ problem

$$y_0^{(4)} = \Lambda_0[(x - \tfrac{1}{2})y_0']', \tag{1.14}$$

$$y_0(0) = y_0'(0) = y_0(1) = y_0'(1) = 0 \tag{1.15a–d}$$

Once again we set $y'''(0) = k$ and $v(x) = y_0'$ and generate

$$v'' = \Lambda_0(x - \tfrac{1}{2})v + k, \tag{1.16}$$

$$v(0) = v(1) = \int_0^1 v(x)dx = 0. \tag{1.17a–c}$$

The relevant eigenvalue problem in this case is

$$\bar{\Phi}'' - \mu(x - \tfrac{1}{2})\bar{\Phi} = 0, \quad \bar{\Phi}(0) = \bar{\Phi}(1) = 0. \tag{*}$$

Clearly

$$\bar{\Phi}_n(x) = C_n\left[A_i(\bar{\mu}_n^{\frac{1}{3}}(x - \tfrac{1}{2})) - \frac{A_i(\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})}{B_i(\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})} B_i(\bar{\mu}_n^{\frac{1}{3}}(x - \tfrac{1}{2}))\right],$$

where A_i and B_i are the standard Airy functions [7] and

$$\frac{A_i(\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})}{B_i(\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})} = \frac{A_i(-\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})}{B_i(-\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}})}$$

defines the eigenvalues $\bar{\mu}_n$. Obviously $\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}}$ is between the n^{th} root of $A_i(-x)$ and the $(n+1)^{\text{st}}$ root of $B_i(-x)$ and approaches the n^{th} root of $A_i(-x)$ for large $\bar{\mu}_n$ [8]. In fact, one can use $A_i(-\frac{1}{2}\bar{\mu}_n^{\frac{1}{3}}) = 0$ to define $\bar{\mu}_n$ with almost no loss in accuracy. The $\bar{\mu}_n$ go symmetrically to $+\infty$ and $-\infty$ and the completeness of $\{\bar{\phi}_n(x)\}$ is as in Case I.

Again it proves convenient to define $w(x) = v(x) - x(x-1)k/2$ so that

$$w'' - \Lambda_0(x - \tfrac{1}{2})w = \Lambda_0(x - \tfrac{1}{2})x(x-1)k/2,$$

$w(0) = w(1) = 0$, $\int_0^1 w(x)dx = k/12$. We expand $w(x)$ and $kx(x-1)/2$ in terms of the eigenfunctions of (*):

$$w(x) = \sum w_n\bar{\phi}_n(x), \quad kx(x-1)/2 = \sum k_n\bar{\phi}_n(x),$$

$$w_n = -\bar{\mu}_n \int_0^1 w(x)(x - \tfrac{1}{2})\bar{\phi}_n(x)dx,$$

$$k_n = -\bar{\mu}_n \int_0^1 \frac{k}{2}x(x-1)(x-\tfrac{1}{2})\bar{\phi}_n(x)dx = \frac{\bar{\mu}_n - \Lambda_0}{\Lambda_0} w_n$$

and, as before,

$$\tfrac{1}{12} = \sum_{-\infty}^{\infty} \frac{\Lambda_0}{\bar{\mu}_n - \Lambda_0}\left(\int_0^1 \bar{\phi}_n(x)dx\right)^2. \tag{E2}$$

Since $\frac{1}{2}\bar{\mu}_1^{\frac{1}{3}} \sim 2.34$ and $\frac{1}{2}\bar{\mu}_2^{\frac{1}{3}} \sim 4.09$ we can conclude that $102.5 < \Lambda_0^{(1)} < 547.3$.

It is also worth mentioning that the large α results obtained here apply equally well for $\nu = 0$ and hence the eigenvalues become asymptotically equal. This conclusion is well borne out in the numerical work of Section 4.

3. Transverse vibrations

We now wish to examine vibrations of the spoke in the plane perpendicular to the plane of rotation. Let $u(s,t)$ denote the transverse displacements, which are assumed sufficiently small that non-linear terms may be consistently neglected. Then $u(s,t)$ is a solution of the dynamic equation.

$$EI\frac{\partial^4 u}{\partial s^4} - \frac{\partial}{\partial s}\left\{T(s)\frac{\partial u}{\partial s}\right\} = -\rho A\frac{\partial^2 u}{\partial t^2},$$

where $T(s)$ is the axial stress given in Equation (1.1). We will seek harmonic vibrations of the form $u(s,t) = w(s)e^{i\omega t}$. Non-dimensionalizing again with $x = s/L$ and $\alpha = R/L$ now leads to the equation

$$\varepsilon_1^3 w^{(4)} - f(x, \alpha)w'' + (x - \alpha)w' - \lambda w = 0, \tag{2.1}$$

where

$$\varepsilon_1^3 = \frac{EI}{\rho A \Omega^2 L^4} \text{ and } \lambda = \left(\frac{\omega}{\Omega}\right)^2.$$

Associated boundary conditions are

$$w(0) = w'(0) = w(1) = w'(1) = 0. \tag{2.2}$$

For rapid rotation rates, ε_1 will be real, positive, and small. Equation (2.1) may thus be examined using singular perturbation methods. The principal sources of mathematical difficulty are now the turning points of the differential equation where $f(x, \alpha)$ vanishes. If $\beta = (\alpha^2 - \alpha + \frac{1}{3})^{\frac{1}{2}}$, these turning points occur at $x = \alpha \pm \beta$. In particular, the turning point at $x_c = \alpha - \beta$ always lies in the interior of the interval $[0, 1]$ for $\alpha \geq 1$. Neither boundary point lies inside the critical layer about x_c, so outer expansions may be consistently used at $x_0 = 0$ and $x_1 = 0$ to form a characteristic equation for the eigenvalues λ. However, appropriate outer expansion valid at x_0 must be continued across the turning point to reach x_1. This is a somewhat delicate matter as, in the complex x-plane, outer expansions for three in any set of four linearly independent solutions of (2.1) will exhibit the Stokes phenomenon. To further complicate the situation, in the complex x-plane, x_1 lies directly on a Stokes line while x_0 lies directly on an anti-Stokes line.

The α dependence in the eigenvalue problem (2.1, 2) may be transfered from the governing differential equation to the boundary conditions by defining the new independent variable

$$y = (x - \alpha)/\beta$$

If $w(x) = w(y)$ and a prime now denotes d/dy, equation (2.1) becomes

$$\varepsilon^3 w^{(4)} - \tfrac{1}{2}(1 - y^2)w'' + yw' - \lambda w = 0 \tag{2.3}$$

where $\varepsilon^3 = \varepsilon_1^3 \beta^{-4}$. The turning point $x_c = \alpha - \beta$ is transformed to $y_c = -1$ and the boundary conditions (2.2) become

$$w(-\alpha/\beta) = w'(-\alpha/\beta) = w\left(\frac{1 - \alpha}{\beta}\right) = w'\left(\frac{1 - \alpha}{\beta}\right) = 0. \tag{2.4}$$

Equation (2.3) arises in the study of the small transverse vibrations of a rotating flexible rod which is clamped on the axis of rotation and has its other end free. Approximations to a set of four linearly independent solutions of (2.3), which are "numerically satisfactory" in the sense of Miller [9], have been obtained by Lakin [2] using the method of matched asymptotic expansions. To write the appropriate outer expansions at $y_0 = -\alpha/\beta$ and $y_1 = (1 - \alpha)/\beta$ in more compact form, let

$$\bar{u}_1(y) = P_\nu(-y) + O(\varepsilon^3),$$

$$\bar{u}_2(y) = -2Q_\nu(-y) + [\log 2 - 2\psi(\nu+1)]P_\nu(-y) + O(\varepsilon^3),$$

$$\bar{v}_\pm(y) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\varepsilon^{-\frac{1}{4}}(y+1)^{\frac{1}{4}}\exp\left\{\pm\varepsilon^{-\frac{1}{2}}\int_{-1}^{y}\left(\frac{1-t^2}{2}\right)^{\frac{1}{2}}dt\right\}\cdot\{1+O(\varepsilon^{\frac{1}{2}})\},$$

$$\bar{v}_1^{(0)} = -\bar{v}_-(y) \text{ and } \bar{v}_2^{(0)} = i\bar{v}_+(y), \tag{2.5a–d}$$

where P_ν and Q_ν are Legendre functions of degree ν and $\psi(z)$ is the digamma function $\Gamma'(z)/\Gamma(z)$. Denote the balanced solution of (2.3) which is regular at y_c by $U_1(y)$, and the balanced-type solution which is balanced in $-2\pi/3 < ph(y+1) < 0$ by $U_2(y)$. Denote the dominant-recessive type solutions of (2.3) which are recessive in $|ph(y+1)| < \pi/3$ and $-\pi < ph(y+1) < -\pi/3$ by $V_1(y)$ and $V_2(y)$. Then, with $\nu(\nu+1)/2 = \lambda$, at y_1 on the Stokes line $ph(y+1) = 0$,

$$\begin{aligned}
&U_1(y_1) \cong \bar{u}_1(y_1),\\
&V_1(y) \cong \bar{v}_1(y_1),\\
&U_2(y_1) \cong \bar{u}_2(y_1) + \pi i\bar{v}_1(y_1),\\
&V_2(y_1) \cong \bar{v}_2(y_1) - \tfrac{1}{2}\bar{u}_1(y_1) - \tfrac{1}{2}\bar{v}_1(y_1).
\end{aligned} \tag{2.6a–d}$$

Similarly, at y_0 on the anti-Stokes line $ph(y+1) = -\pi$,

$$\begin{aligned}
&U_1(y_0) = \bar{u}_1(y_0),\\
&V_2(y_0) = \bar{v}_2(y_0),\\
&U_2(y_0) = \bar{u}_2(y_0) - 2\pi i\bar{v}_2(y_0),\\
&V_1(y_0) = \bar{v}_1(y_0) - \bar{v}_2(y_0) - \bar{u}_1(y_0).
\end{aligned} \tag{2.7a–d}$$

The exponentials in $\bar{v}_1$ and $\bar{v}_2$ are maximally recessive and dominant, respectively, at y_1, while $\bar{u}_1$ and $\bar{u}_2$ are both balanced at y_1. Approximations to all four solutions are balanced at y_0.

The exact eigenvalue relationship obtained from the boundary conditions (2.4) involves a four-by-four determinant. If $W_k(X, Y)$ denotes the Wronskian of $X(y)$ and $Y(y)$ at y_k ($k = 0, 1$), without approximation

$$\begin{aligned}
\Delta = {} & W_0(U_1, U_2)W_1(V_1, V_2) - W_0(U_1, V_1)W_1(U_2, V_2) + W_0(U_1, V_2)W_1(U_2, V_1)\\
& + W_0(U_2, V_1)W_1(U_1, V_2) - W_0(U_2, V_2)W_1(U_1, V_1) + W_0(V_1, V_2)W_1(U_1, U_2) = 0.
\end{aligned}$$

Using (2.7) and (2.8) in the Wronskians above gives a consistent approximation to Δ which contains three types of terms and may be written as $\Delta = \mathscr{D} + \mathscr{B} + \mathscr{R}$. Terms in $\mathscr{D}$ contain the positive exponential in $\bar{v}_+(y_1)$ and are maximally dominant at y_1. Terms in $\mathscr{B}$ and $\mathscr{R}$ are balanced and recessive. Lowest order terms in $\mathscr{D}$ will thus give first approximation to the desired eigenvalues. In particular, $\lambda^{(0)}$ is a root of the equation

$$\left\{P_\nu(\alpha/\beta)Q_\nu\left(\frac{\alpha-1}{\beta}\right) - P_\nu\left(\frac{\alpha-1}{\beta}\right)Q_\nu(\alpha/\beta)\right\}\sin\zeta(\alpha,\varepsilon)$$
$$+\frac{\pi}{2}P_\nu(\alpha/\beta)P_\nu\left(\frac{\alpha-1}{\beta}\right)\exp\{i\zeta(\alpha,\varepsilon)\} = 0, \tag{2.8}$$

where

$$\zeta(\alpha,\varepsilon) = (2\varepsilon)^{-\frac{3}{2}}\left\{\frac{\alpha}{\beta}\sqrt{\alpha-\tfrac{1}{3}} + \log\left(\frac{\alpha-\sqrt{\alpha-\frac{1}{3}}}{\beta}\right)\right\}$$

and

$$\lambda^{(0)} = \nu(\nu+1)/2.$$

If $\zeta(\alpha,\varepsilon) \neq m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ the eigenvalue relationship (2.8) will be irreducibly complex. In this case, the boundary value problem (2.3, 4) will not have real, positive eigenvalues and the corresponding frequencies ω, involving $\lambda^{\frac{1}{2}}$, will have negative imaginary parts. Hence, for rapid rotation rates the harmonic vibrations $we^{i\omega t}$ will grow with time, i.e. they will be unstable. A similar conclusion holds even in the special case $\zeta = m\pi$ when equation (2.8) is real. Here, $P_\nu(\alpha/\beta) = 0$ with $\alpha/\beta > 1$ implies that P_ν must be a conical function so that $\nu_n = -\frac{1}{2} + iK_n$, $n = 0, 1, 2, \ldots$, with $K_n > 0$. This situation is similar to an off-clamped rod with one free end studied by Lakin and Nachman [5]. To lowest order the eigenvalues are $\lambda_n^{(0)} = -(K_n^2 + \frac{1}{4})$, real but purely negative. Hence, the vibrations are again unstable.

This work has not explicitly treated non-transverse vibrations of the rotating spoke. However, experience in [5] indicates that transverse vibrations are more stable than vibrations in other planes. Hence, instability may also be expected for the non-transverse vibrations.

4. Approximation of lowest eigenvalues by the Galerkin method

Consider transverse vibrations of a spoke:

$$L(w) = \varepsilon^3 w^{\text{iv}} + \tfrac{1}{2}(x^2 - 2\alpha x + \alpha - \tfrac{1}{3})w'' + (x-\alpha)w' - \lambda w = 0, \tag{3.1}$$

where $\alpha \geq 1$, $0 < \varepsilon \ll 1$ and w satisfies the boundary conditions

$$w(0) = w'(0) = w(1) = w'(1) = 0. \tag{3.2}$$

The solution will be approximated by a linear combination

$$w_n = \sum_{k=1}^{n} C_k\phi_k(x) \tag{3.3}$$

of basis functions ϕ_k satisfying the conditions (3.2). Assuming that $\{\phi_1, \phi_2, \ldots\}$ is a complete system of functions in a space containing the solution of (3.1) the Galerkin approximation can be found from the following orthogonality conditions

$$\int_0^1 L(w_n)\phi_i dx = 0, \qquad i = 1, \ldots, n. \tag{3.4}$$

The resulting homogeneous system of linear equations

$$Ac = \sum_{j=1}^{n} (a_{ij} - \lambda b_{ij})c_j = 0, \qquad i = 1, \ldots, n, \tag{3.5}$$

where

$$a_{ij} = \int_0^1 [\varepsilon^3\phi_j^{\text{iv}} + \tfrac{1}{2}(x^2 - 2\alpha x + \alpha - \tfrac{1}{3})\phi_j'' + (x - \alpha)\phi_j']\phi_i dx, \tag{3.6}$$

$$b_{ij} = \int_0^1 \phi_j\phi_i dx,$$

has a nontrivial solution only if the determinant of A vanishes

$$\det(A) = 0. \tag{3.7}$$

Then the roots $\lambda_1, \ldots, \lambda_n$ of (3.7) will approximate the first n eigenvalues of (3.1) corresponding to the eigenfunctions

$$w_{nl} = \sum_{k=1}^{n} C_{kl}\phi_k(x). \tag{3.8}$$

In order to approximate the lowest eigenvalues of (3.1) subject to (3.2) let

$$w_n(x) = x^2(1 - x)^2(c_1 + c_2 x + \ldots + c_n x^n). \tag{3.9}$$

Then the second approximation $w_2(x)$ yields

$$\begin{aligned} &\left(\frac{4}{5}\varepsilon^3 - \frac{\lambda}{630}\right)C_1 + \left(\frac{2}{5}\varepsilon^3 + \frac{2\alpha - 1}{2520} - \frac{\lambda}{1260}\right)C_2 = 0, \\ &\left(\frac{2}{5}\varepsilon^3 + \frac{2\alpha - 1}{2520} - \frac{\lambda}{1260}\right)C_1 + \left(\frac{12}{35}\varepsilon^3 + \frac{33\alpha - 16}{41580} - \frac{\lambda}{2310}\right)C_2 = 0. \end{aligned} \tag{3.10}$$

Equating the determinant to zero we obtain a quadratic equation in λ

$$\lambda^2 - (4464\varepsilon^3 + \tfrac{1}{3})\lambda + 1995840\varepsilon^6 + 168\varepsilon^3 - \tfrac{11}{4}(2\alpha - 1)^2 = 0, \tag{3.11}$$

the solution of which is

$$\lambda_{1,2} = 2232\varepsilon^3 + \tfrac{1}{6} \mp \sqrt{(1728\varepsilon^3 + \tfrac{1}{6})^2 + 11(\alpha - \tfrac{1}{2})^2}. \tag{3.12}$$

Then the eigenfunction corresponding to the smallest eigenvalue λ_1 is given by

$$w_{21}(x) = x^2(1-x)^2\left(1 - 2\frac{\sqrt{11\beta^2+1}-1}{\sqrt{11\beta^2+1}+\beta-1}x\right), \tag{3.13}$$

where $\beta = (6\alpha - 3)/(10368\varepsilon^3 + 1)$.

From (3.11) or (3.12) it is possible to estimate the critical $\varepsilon = \varepsilon_c$ for which $\lambda_1(\alpha, \varepsilon_c) = 0$:

$$\varepsilon_c^3 = \left(\sqrt{\frac{5}{7}\left(\alpha - \frac{1}{2}\right)^2 + \left(\frac{1}{66}\right)^2} - \frac{1}{66}\right)\Big/360. \tag{3.14}$$

Its inverse $\lambda_c = \varepsilon_c^{-3}$ corresponds to the lowest positive eigenvalue for transverse buckling of the spoke. The eigenfunction is given by

$$w_{2c} = x^2(1-x)^2\left(1 - \frac{2016\varepsilon_c^3}{1008\varepsilon_c^3 + 2\alpha - 1}x\right). \tag{3.15}$$

Finally, equation (3.11) can be used to estimate the lowest positive eigenvalue λ_I for the in-plane buckling of the spoke, by setting $\lambda = 1$. Then

$$\lambda_I^{-1} = \varepsilon_I^3 = \left(\sqrt{\frac{5}{7}\left(\alpha - \frac{1}{2}\right)^2 + \frac{22801}{213444}} + \frac{179}{462}\right)\Big/360 \tag{3.16}$$

and the corresponding eigenfunction

$$w_{2I} = x^2(1-x)^2\left(1 - \frac{2016\varepsilon_I^3 - 4}{1008\varepsilon_I^3 + 2\alpha - 3}x\right). \tag{3.17}$$

An accurate approximation to the lowest positive eigenvalues and eigenfunctions can be obtained by employing piecewise polynomial basis functions.

For cubic-Hermite polynomials we may write (3.8) in the form

$$w_{2n-2}(x) = \sum_{j=1}^{n-1} [c_k\psi_0(nx-j) + d_k\psi_1(nx-j)], \quad 0 \le x \le 1, \tag{3.18}$$

where

$$\psi_0(x) = \begin{cases} 0, & x > 1 \\ (1-x)^2(1+2x), & 0 \le x \le 1. \\ \psi_0(-x), & x < 0 \end{cases}$$
$$\psi_1(x) = \begin{cases} 0, & x > 1 \\ (1-x)^2 x, & 0 \le x \le 1. \\ -\psi_1(-x), & x < 0 \end{cases} \tag{3.19}$$

The error in the approximation is of the order $O(n^{-4})$. For $n = 40$ the eigenvalues λ_c and λ_I are accurate to 4 significant figures.

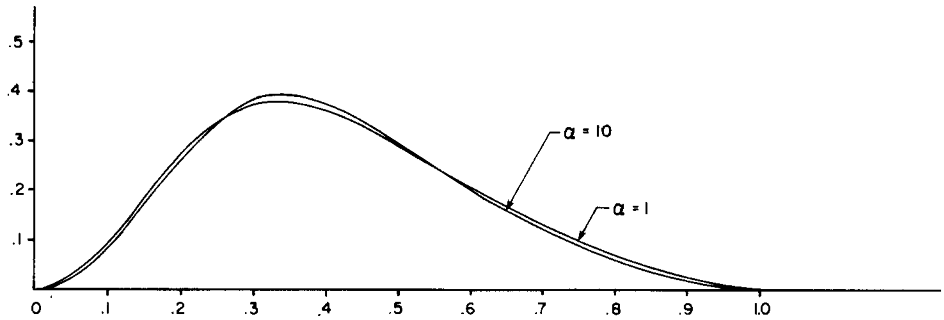

Figure 3.

α	1	2	10	100
λ_c (transverse)	681.3	233.9	37.18	3.552
λ_I (in-plane)	371.2	188.9	35.96	3.541

This agrees fairly well with (3.14) and (3.16) which yield values about 20% larger than those listed in our table. The eigenfunctions corresponding to $\alpha = 1$, 10 are depicted in Figures 3 and 4. (All eigenfunctions are of norm 1 in L_2).

We complete the table by presenting the second and third eigenvalues for transverse and in-plane buckling.

Second Eigenvalue:

α	1	2	10	100
λ_c (transverse)	1777.3	569.16	87.51	8.31
λ_I (in-plane	1400.49	523.66	86.35	8.30

Third Eigenvalue:

α	1	2	10	100
λ_c (transverse)	4336.5	1378.8	211.65	20.10
λ_I (in-plane)	3323.23	1256.04	208.57	20.08

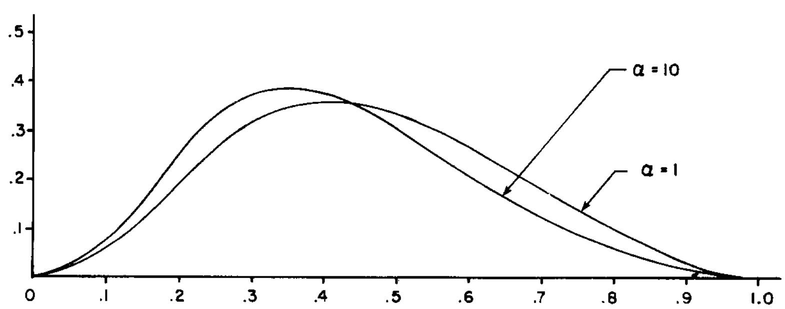

Figure 4.

5. Conclusion

The buckling and vibration of a rapidly rotating spoke was investigated. Bounds for the eigenvalues of the buckled modes were obtained for all rotation speeds and eigenfrequencies of vibration for high rotation speeds. Also a table of the first three buckling eigenvalues for various dimensionless lengths was provided.

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